

On the Unitarity of D=9,10,11 Conformal Supersymmetry

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We consider the unitarity of $D = 9, 10, 11$ conformal supersymmetry using the recently established classification of the UIRs of the superalgebras $osp(1|2n, \mathbb{R})$.

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1 Introduction

Recently, applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. In these applications most prominent role play the superalgebras $osp(1|2n)$, cf., e.g., [1–10]. Initially, the superalgebra $osp(1|32)$ was put forward for $D = 10$ [1]. Later it was realized that $osp(1|2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, resp.) [7]. In all cases one needs to find first the UIRs of $osp(1|2n, \mathbb{R})$. This was done for general n in [11]. In the present paper we consider the implications for conformal supersymmetry for $D = 9, 10, 11$.

2 UIRs of the superalgebras $osp(1|2n, \mathbb{R})$

The conformal superalgebras in $D = 9, 10, 11$ are $\mathfrak{g} = osp(1|2n, \mathbb{R})$, $n = 16, 16, 32$, resp., cf. [1, 7]. The even subalgebra of $osp(1|2n, \mathbb{R})$ is the algebra $sp(2n, \mathbb{R})$

with maximal compact subalgebra $\mathfrak{k} = u(n) \cong su(n) \oplus u(1)$. We label the relevant representations of \mathfrak{g} by the signature:

$$\chi = [d; a_1, \dots, a_{n-1}] \quad (1)$$

where d is the conformal weight, and a_1, \dots, a_{n-1} are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of \mathfrak{k}). The positive energy UIRs of \mathfrak{g} for any $n > 1$ are given in the following list [11]:

$$\begin{aligned} d &\geq d_1 = n - 1 + \frac{1}{2}(a_1 + \dots + a_{n-1}), \quad \text{no restrictions on } a_j, \\ d &= d_{12} = n - 2 + \frac{1}{2}(a_2 + \dots + a_{n-1} + 1), \quad a_1 = 0, \\ &\dots \\ d &= d_{j-1,j} = n - j + \frac{1}{2}(a_j + \dots + a_{n-1} + 1), \quad a_1 = \dots = a_{j-1} = 0, \\ &\dots \\ d &= d_{n-1,n} = \frac{1}{2}, \quad a_1 = \dots = a_{n-1} = 0. \end{aligned} \quad (2)$$

These UIRs are realized as the irreducible quotients of Verma modules V^Λ of lowest weight $\Lambda = \Lambda(\chi)$. The weight is fixed by its products with the simple roots α_i ($i = 1, \dots, n$) of \mathfrak{g} [11]:

$$(\Lambda, \alpha_k^\vee) = (\Lambda, 2\alpha_k/(\alpha_k, \alpha_k)) = (\Lambda, \alpha_k) = -a_k, \quad k < n, \quad (3a)$$

$$(\Lambda, \alpha_n^\vee) = 2(\Lambda, \alpha_n) = 2d + a_1 + \dots + a_{n-1}. \quad (3b)$$

3 Unitarity for the conformal subalgebras

From the prescription of [7] follows that the even subalgebra $sp(2n, \mathbb{R})$, $n = 16, 16, 32$, resp., contains the conformal algebra $\mathfrak{C} = so(D, 2)$, $D = 9, 10, 11$. Then \mathfrak{k} contains the maximal compact subalgebra $so(D) \oplus so(2)$ of \mathfrak{C} , $so(2)$ being identified with the $u(1)$ factor of \mathfrak{k} , and $su(n)$ contains the algebra $so(D)$. The easiest way to describe the embeddings is via the root systems. The superalgebra \mathfrak{g} is the split real form of the basic classical superalgebra $osp(1|2n)$ and has the same root system. The root system of $su(n)$, actually of $sl(n)$, is comprised of the even simple roots of \mathfrak{g} : $\alpha_i, i = 1, \dots, n-1$, with standard non-zero products: $(\alpha_i, \alpha_i) = 2$, ($i = 1, \dots, n-1$), $(\alpha_i, \alpha_{i+1}) = -1$, ($i = 1, \dots, n-2$). The root system of $so(D)$ is comprised of simple roots $\gamma_j, j = 1, \dots, \ell \equiv [D/2]$.

For even D the non-zero scalar products are: $(\gamma_j, \gamma_j) = 2\kappa$, ($j = 1, \dots, \ell$), $(\gamma_j, \gamma_{j+1}) = -\kappa$, ($j = 1, \dots, \ell-2$), $(\gamma_{\ell-2}, \gamma_\ell) = -\kappa$, where κ is a non-zero common multiple (it is inessential since it cancels in the Cartan matrix elements). In the case of interest $D = 10$, $\ell = 5$, $n = 16$:

$$\begin{aligned} \gamma_1 &= \alpha_4 + \alpha_7 + \alpha_9 + \alpha_{12}, \\ \gamma_2 &= \alpha_3 + \alpha_6 + \alpha_{10} + \alpha_{13}, \\ \gamma_3 &= \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12} + \alpha_{14}, \\ \gamma_4 &= \alpha_3 + \alpha_5 + \alpha_7 + \alpha_{15}, \end{aligned}$$

$$\gamma_5 = \alpha_1 + \alpha_9 + \alpha_{11} + \alpha_{13} . \quad (4)$$

(The roots γ_i satisfy the prescribed products with $\kappa = 4$.) Correspondingly, the $so(10)$ Dynkin labels $r_k \equiv -\kappa(\Lambda, \gamma_k^\vee) = -(\Lambda, \gamma_k)$ are:

$$\begin{aligned} r_1 &= a_4 + a_7 + a_9 + a_{12} , \\ r_2 &= a_3 + a_6 + a_{10} + a_{13} , \\ r_3 &= a_2 + a_4 + a_5 + a_6 + a_7 + 2a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{14} , \\ r_4 &= a_3 + a_5 + a_7 + a_{15} , \\ r_5 &= a_1 + a_9 + a_{11} + a_{13} . \end{aligned} \quad (5)$$

The dimensions of the $so(10)$ UIRs is:

$$\dim L(r_1, \dots, r_5) = \prod_{1 \leq s < t \leq 5} \frac{n_t^2 - n_s^2}{(t-s)(t+s-2)} , \quad (6)$$

where we use the additional parametrization:

$$\begin{aligned} n_1 &= \frac{1}{2}(r_5 - r_4) , & n_2 &= \frac{1}{2}(r_5 + r_4) + 1 \\ n_s &= \frac{1}{2}(r_5 + r_4) + r_3 + \dots + r_{6-s} + s - 1 , & s &= 3, \dots, 5 . \end{aligned} \quad (7)$$

The parameters n_s are either all integer or all half-integer obeying:

$$n_5 > n_4 > n_3 > n_2 > |n_1| \geq 0 . \quad (8)$$

It is known that the unitarity restrictions for a conformal superalgebra \mathfrak{a} are stronger than those for the even subalgebra of \mathfrak{a} (for $D = 4, 6$ cf. [12]). Here, in addition the conformal algebra $so(D, 2)$ is smaller than the even subalgebra $sp(2n)$. Thus, the unitarity conditions for $so(D, 2)$ are given only in terms of r_i . Thus, $so(D, 2)$ unitarity would not require that all parameters a_k are non-negative integers - that would be required only for their combinations r_i . The unrestricted parameters a_k are combined in so-called tensorial charges [1]. We shall leave this together with more detailed analysis for a follow-up paper.

Here, due to the lack of space, we only consider briefly the reduction of the the fundamental irreps of $su(n)$ to $so(D)$ irreps. We list only the main $so(D)$ component with signature following directly from the embedding formulae, e.g., (5) for $so(10)$.

Of course, the one-dimensional irreps, when $a_i = 0 = r_s$ for all i, s , coincide. In the Dynkin labeling of the $n-1$ fundamental irreps Λ_k of $su(n)$ are characterized for fixed k by $a_i = \delta_{ik}$. The fundamental irreps Λ_t^o of $so(D)$ are characterized for fixed k by $r_s = \delta_{st}$. The 16-dimensional fundamental $su(16)$ UIR with $a_1 = 1$ gives the fundamental 16-dimensional $so(10)$ spinor when $r_5 = 1$, while the conjugated 16-dimensional fundamental $su(16)$ UIR with $a_{15} = 1$ gives the conjugated 16-dimensional $so(10)$ spinor with $r_4 = 1$. We summarise the results in

the following table:

Λ_i	$\dim(\Lambda_i)$	$\chi = [r_1, r_2, r_3, r_4, r_5]$	$\dim(\chi)$	
Λ_1	16	$[0, 0, 0, 0, 1]$	16	
Λ_{15}	16	$[0, 0, 0, 1, 0]$	16	
Λ_2, Λ_{14}	120	$[0, 0, 1, 0, 0]$	120	
Λ_3	560	$[0, 1, 0, 1, 0]$	560	
Λ_{13}	560	$[0, 1, 0, 0, 1]$	560	
Λ_4, Λ_{12}	1820	$[1, 0, 1, 0, 0]$	945	(9)
Λ_5	4368	$[0, 0, 1, 1, 0]$	1200	
Λ_{11}	4368	$[0, 0, 1, 0, 1]$	1200	
Λ_6, Λ_{10}	8008	$[0, 1, 1, 0, 0]$	2970	
Λ_7	11440	$[1, 0, 1, 1, 0]$	8800	
Λ_9	11440	$[1, 0, 1, 0, 1]$	8800	
Λ_8	12870	$[0, 0, 2, 0, 0]$	4125	

Part of the above analysis is done in the oscillator approach in [4].

For odd D the non-zero scalar products are: $(\gamma_j, \gamma_j) = 2\kappa$, $(j = 1, \dots, \ell - 1)$,
 $(\gamma_\ell, \gamma_\ell) = \kappa$, $(\gamma_j, \gamma_{j+1}) = -\kappa$, $(j = 1, \dots, \ell - 1)$.

In the case $D = 9$, $\ell = 4$, $n = 16$, we have:

$$\begin{aligned}
 \gamma_1 &= \alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_9 + 2\alpha_{10} + \alpha_{11} , \\
 \gamma_2 &= \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_{11} + 2\alpha_{12} + \alpha_{13} , \\
 \gamma_3 &= \alpha_2 + \alpha_6 + \alpha_7 + 2\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{14} , \\
 \gamma_4 &= \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 + \alpha_9 + \alpha_{11} + \alpha_{13} + \alpha_{15}) ,
 \end{aligned} \tag{10}$$

(with $\kappa = 4$). Then, the $so(9)$ Dynkin labels and dimensions are:

$$\begin{aligned}
 r_1 &= a_5 + 2a_6 + a_7 + a_9 + 2a_{10} + a_{11} , \\
 r_2 &= a_3 + 2a_4 + a_5 + a_{11} + 2a_{12} + a_{13} , \\
 r_3 &= a_2 + a_6 + a_7 + 2a_8 + a_9 + a_{10} + a_{14} , \\
 r_4 &= a_1 + a_3 + a_5 + a_7 + a_9 + a_{11} + a_{13} + a_{15} ,
 \end{aligned} \tag{11}$$

$$\dim L(r_1, \dots, r_4) = \frac{2^4 n_1 n_2 n_3 n_4}{7.5.3} \prod_{1 \leq s < t \leq 4} \frac{n_t^2 - n_s^2}{(t-s)(t+s-1)} , \tag{12}$$

$$n_1 = \frac{1}{2}(r_4 + 1) , \quad n_s = \frac{1}{2}(r_4 - 1) + r_3 + \dots + r_{5-s} + s , \quad s = 2, 3, 4, \tag{13}$$

$$n_4 > n_3 > n_2 > n_1 > 0 . \tag{14}$$

The fundamental 16-dimensional $so(9)$ spinor - obtained for $r_4 = 1$ - is contained without reduction in both conjugated 16-dimensional fundamental $su(16)$ UIRs

with $a_1 = 1$ and $a_{15} = 1$. We summarise the results in the following table:

Λ_i	$\dim(\Lambda_i)$	$\chi = [r_1, r_2, r_3, r_4]$	$\dim(\chi)$
Λ_1, Λ_{15}	16	$[0, 0, 0, 1]$	16
Λ_2, Λ_{14}	120	$[0, 0, 1, 0]$	84
Λ_3, Λ_{13}	560	$[0, 1, 0, 1]$	432
Λ_4, Λ_{12}	1820	$[0, 2, 0, 0]$	495
Λ_5, Λ_{11}	4368	$[1, 1, 0, 1]$	2560
Λ_6, Λ_{10}	8008	$[2, 0, 1, 0]$	2457
Λ_7, Λ_9	11440	$[1, 0, 1, 1]$	5040
Λ_8	12870	$[0, 0, 2, 0]$	1980

(15)

In the case $D = 11$, $\ell = 5$, $n = 32$, we have:

$$\begin{aligned}
 \gamma_1 &= \alpha_5 + \alpha_8 + \alpha_{10} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \\
 &\quad + \alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{22} + \alpha_{24} + \alpha_{27} , \\
 \gamma_2 &= \alpha_4 + \alpha_7 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12} + \alpha_{14} + 2\alpha_{15} + 2\alpha_{16} + \\
 &\quad + 2\alpha_{17} + \alpha_{18} + \alpha_{20} + \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{25} + \alpha_{28} , \\
 \gamma_3 &= \alpha_3 + \alpha_5 + \alpha_6 + \alpha_8 + \alpha_9 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + 2\alpha_{16} + \\
 &\quad + \alpha_{17} + \alpha_{18} + \alpha_{19} + \alpha_{20} + \alpha_{23} + \alpha_{24} + \alpha_{26} + \alpha_{27} + \alpha_{29} , \\
 \gamma_4 &= \alpha_2 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_{10} + \alpha_{11} + \alpha_{12} + \alpha_{13} + \alpha_{14} + \\
 &\quad + \alpha_{18} + \alpha_{19} + \alpha_{20} + \alpha_{21} + \alpha_{22} + \alpha_{24} + \alpha_{25} + \alpha_{26} + \alpha_{28} + \alpha_{30} , \\
 \gamma_5 &= \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 + \alpha_9 + \alpha_{11} + \alpha_{13} + \alpha_{15} + \\
 &\quad + \alpha_{17} + \alpha_{19} + \alpha_{21} + \alpha_{23} + \alpha_{25} + \alpha_{27} + \alpha_{29} + \alpha_{31}) ,
 \end{aligned} \tag{16}$$

(with $\kappa = 8$). Then, the $so(11)$ Dynkin labels and dimensions are:

$$\begin{aligned}
 r_1 &= a_5 + a_8 + a_{10} + a_{13} + a_{14} + a_{15} + \\
 &\quad + a_{17} + a_{18} + a_{19} + a_{22} + a_{24} + a_{27} , \\
 r_2 &= a_4 + a_7 + a_9 + a_{10} + a_{11} + a_{12} + a_{14} + 2a_{15} + 2a_{16} + \\
 &\quad + 2a_{17} + a_{18} + a_{20} + a_{21} + a_{22} + a_{23} + a_{25} + a_{28} , \\
 r_3 &= a_3 + a_5 + a_6 + a_8 + a_9 + a_{12} + a_{13} + a_{14} + a_{15} + 2a_{16} + \\
 &\quad + a_{17} + a_{18} + a_{19} + a_{20} + a_{23} + a_{24} + a_{26} + a_{27} + a_{29} , \\
 r_4 &= a_2 + a_4 + a_6 + a_7 + a_8 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + \\
 &\quad + a_{18} + a_{19} + a_{20} + a_{21} + a_{22} + a_{24} + a_{25} + a_{26} + a_{28} + a_{30} , \\
 r_5 &= a_1 + a_3 + a_5 + a_7 + a_9 + a_{11} + a_{13} + a_{15} + \\
 &\quad + a_{17} + a_{19} + a_{21} + a_{23} + a_{25} + a_{27} + a_{29} + a_{31} ,
 \end{aligned} \tag{17}$$

$$\dim L(r_1, \dots, r_5) = \frac{2^5 n_1 n_2 n_3 n_4 n_5}{9 \cdot 7 \cdot 5 \cdot 3} \prod_{1 \leq s < t \leq 5} \frac{n_t^2 - n_s^2}{(t-s)(t+s-1)} , \tag{18}$$

$$n_1 = \frac{1}{2}(r_5 + 1) , \quad n_s = \frac{1}{2}(r_5 - 1) + r_4 + \dots + r_{6-s} + s , \quad s = 2, \dots, \ell, \tag{19}$$

$$n_5 > n_4 > n_3 > n_2 > n_1 > 0 . \tag{20}$$

The fundamental 32-dimensional $so(11)$ spinor - obtained for $r_5 = 1$ - is contained without reduction in both 32-dimensional fundamental $su(32)$ UIRs with $a_1 = 1$ and $a_{31} = 1$. We summarise the results in the following table:

Λ_i	$\dim(\Lambda_i)$	$\chi = [r_1, r_2, r_3, r_4, r_5]$	$\dim(\chi)$	
Λ_1, Λ_{31}	32	$[0, 0, 0, 0, 1]$	32	
Λ_2, Λ_{30}	496	$[0, 0, 0, 1, 0]$	330	
Λ_3, Λ_{29}	4960	$[0, 0, 1, 0, 1]$	3520	
Λ_4, Λ_{28}	35960	$[0, 1, 0, 1, 0]$	11583	
Λ_5, Λ_{27}	201376	$[1, 0, 1, 0, 1]$	28512	
Λ_6, Λ_{26}	906192	$[0, 0, 1, 1, 0]$	23595	
Λ_7, Λ_{25}	3365856	$[0, 1, 0, 1, 1]$	160160	
Λ_8, Λ_{24}	10518300	$[1, 0, 1, 1, 0]$	178750	(21)
Λ_9, Λ_{23}	28048800	$[0, 1, 1, 0, 1]$	91520	
$\Lambda_{10}, \Lambda_{22}$	64512240	$[1, 1, 0, 1, 0]$	78650	
$\Lambda_{11}, \Lambda_{21}$	129024480	$[0, 1, 0, 1, 1]$	160160	
$\Lambda_{12}, \Lambda_{20}$	225792840	$[0, 1, 1, 1, 0]$	525525	
$\Lambda_{13}, \Lambda_{19}$	347373600	$[1, 0, 1, 1, 1]$	2114112	
$\Lambda_{14}, \Lambda_{18}$	471435600	$[1, 1, 1, 1, 0]$	3128697	
$\Lambda_{15}, \Lambda_{17}$	565722720	$[1, 2, 1, 0, 1]$	6040320	
Λ_{16}	601080390	$[0, 2, 2, 0, 0]$	1718496	

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